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98%-EFFECTIVE INTEGER-RATIO LOT-SIZING FOR ONE-WAREHOUSE MULTI-RETAILER SYSTEMS*

ROBIN ROUNDY
School of Operations Research and Industrial Engineering, Cornell University, Ithaca, New York 14853

A warehouse supplies N retailers. Constant external demand occurs at each retailer, and shortages are not allowed. There are linear holding costs and fixed costs for ordering at the warehouse and at each retailer. The goal is to minimize the long-run average cost over an infinite time horizon.

We define a new class of policies for this problem whose simple structure facilitates both computation and implementation. The cost of a policy that is optimal within this class is shown to be within 2% of the cost of an optimal policy for the original problem, in the worst case. Such a policy can be computed in $O(N \log N)$ time.
(INVENTORY/PRODUCTION; DETERMINISTIC MODELS; MULTI-ECHELON; ORDERING POLICIES)

1. Introduction

Consider a warehouse that is the sole supplier of N retailers. Customer demand occurs at each retailer at a constant rate. This demand must be met as it occurs over an infinite horizon without shortages or backlogging. Orders placed by retailers generate demands at the warehouse. There is a holding cost rate per unit stored per unit time and a fixed charge for each order placed at the warehouse and at each retailer. The demand rates, holding cost rates, and setup costs are stationary and facility dependent. Delivery of orders is assumed to be instantaneous. The goal is to find a policy with minimum or near-minimum long-run average cost. There is no known method for solving the continuous-time version of this problem, or for solving the discrete-time finite-horizon version in polynomial time.

Optimal policies can be very complex, and this complexity would make them unattractive even if they could be computed efficiently. Typically an optimal policy entails nonstationary order quantities and order intervals, i.e., the order quantities and the times between successive orders vary over time at each facility. The best that can be hoped is that the order quantities and intervals at each facility in the system are periodic (Graves and Schwarz 1977). However the periods of such policies can be so long that they include an arbitrarily large number of orders at each of the facilities in the system (Roundy 1983).

Since optimal policies are difficult to find and are often very complex, we are led to seek simple policies with guaranteed high effectiveness. The *effectiveness* of a policy is 100% times the ratio of the infimum of the average cost over all policies to the average cost of the policy in question.

In what follows we introduce two simple policies, called *q-optimal integer-ratio* and *optimal power-of-two* policies. We prove that for any data set, the effectiveness of q-optimal integer-ratio and optimal power-of-two policies are at least 94% and 98%, respectively. Though the latter policy has higher guaranteed effectiveness, the former has the advantage of giving the user considerable freedom in determining the times at which orders will be placed. Both policies can be computed in $O(N \log N)$ time. Moreover, they are simple and easy to implement.

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The remainder of this section is devoted to reviewing and discussing earlier work on this and related problems. In §2 we introduce the class of integer-ratio policies. In §3 we find a lower bound on the long-run average cost of all feasible policies. In §4 we show that the lower bound is closely related to the average cost of order-preserving integer-ratio policies, making their effectiveness easy to evaluate. The proof of the Lower Bound Theorem is given in §5. We discuss q-optimal integer-ratio and optimal power-of-two policies in §§6 and 7, respectively. In §8 we draw our conclusions.

Review of the Literature

The best algorithms for the nonstationary discrete-time finite-horizon one-warehouse multi-retailer lot-sizing problem are Veinott's (1969) generalization of Zangwill's algorithm for series systems (1969) to arborescence distribution systems, and an algorithm of Kalymon (1972). Unfortunately both of these algorithms have exponential running times. If either the number of time periods or the number of retailers is small then one of these algorithms will probably be quite efficient, but neither is likely to be useful when both of these parameters are larger than about 15.

An optimal policy for the stationary continuous-time infinite-horizon one-warehouse multi-retailer lot-sizing problem has not been found. However Schwarz (1973) has proven that if an optimal schedule exists, there is one with the following properties:

Zero-Inventory Ordering. Each facility (i.e., warehouse or retailer) orders only when its inventory is zero.

Last-Minute Ordering. The warehouse orders only when at least one retailer orders.

Stationarity-Between-Orders. At each retailer all orders placed between two successive orders at the warehouse are of equal size.

For the finite-horizon discrete-time case the zero-inventory-ordering property was proven by Wagner and Whitin (1958) for a single facility and by Zangwill (1966) and Veinott (1969) for many facilities. The stationarity-between-orders property generalizes a finite-horizon, single-facility result of Carr and Howe (4).

A number of authors (Taha and Skieth 1970, Jensen and Khan 1972, Crowston, Wagner, and Henshaw 1972, Crowston, Wagner, and Williams 1973, Schwarz 1973, Schwarz and Schrage 1975, Graves and Schwarz 1977, Bigham and Mogg 1977, Singer 1978, Szendrovits and Goyal 1978, Bigham and Mogg 1979, Williams 1981, 1982, Maxwell and Muckstadt 1983, Jackson, Maxwell, and Muckstadt 1984) have considered the possibility of restricting the class of admissible policies for stationary continuous-time infinite-horizon multi-stage production/inventory problems with the goal of finding a good approximation to optimal policies without excessive computation. While these papers deal with many different configurations of systems (series, assembly, etc.), they all make similar assumptions about the types of policies they allow. A policy is called *stationary* if each facility orders at equally-spaced points in time and in equal amounts. A policy is *nested* if each facility orders every time any of its immediate suppliers does, and perhaps at other times as well (Veinott 1969). Policies that are both stationary and nested are called *stationary-nested* or *single-cycle*. Nearly all of these papers restrict attention to stationary-nested policies.

The work of Jackson, Maxwell, and Muckstadt is especially relevant. In (1983), Maxwell and Muckstadt developed a heuristic for computing stationary nested policies in very complex, multi-stage, multi-product systems. In (1984), Jackson, Maxwell, and Muckstadt developed a heuristic for the joint replenishment problem, which may be viewed as a special case of the one-warehouse, multi-retailer system (Graves 1979, Roundy 1983). These heuristics, which were developed independently of this work, are very similar to ours. After learning of this work and of the results in Roundy (1984), they showed that the worst-case effectiveness of their heuristic for the joint replenishment problem is 94% and that the average cost of the heuristic in Maxwell and

TABLE 1
Problem Data

	Warehouse	Retailer One	Retailer Two
Set-up Cost	1	1	K
Demand Rate	—	2	d
Holding Cost Rate	1	2	2

TABLE 2
Upper Bounds on Effectiveness of Optimal Nested Policies

d	K = 2	32	∞
1/2	96%	75%	58%
1/32	82%	44%	17%
0	73%	29%	0%

Muckstadt (1983) is always within 6% of the average cost of an optimal nested policy.

Unfortunately the effectiveness of an optimal nested policy can be arbitrarily close to zero (Roundy 1983). Consider a warehouse that supplies two retailers, one of which has a relatively large setup cost K and a relatively small demand rate d. The data for the problem are given in Table 1.

Because K is large and d is small, one would expect retailer two to order less often than the warehouse. However in a nested policy retailer two must order every time the warehouse does. This causes nested policies to have low effectiveness. Table 2, taken from Roundy (1983), gives very tight upper bounds on the effectiveness of optimal nested policies for a few values of K and d.

In this model two retailers with the same setup cost, holding cost rate, and demand rate will order simultaneously. Therefore a number of low-demand retailers with moderate setup costs can have the same effect as a single low-demand retailer with a large setup cost.

In summary optimal policies for this problem seem to be very difficult to compute in either discrete or continuous time, and nested policies can have very low effectiveness in the worst case. Thus a more appropriate set of restrictions on the class of admissible policies seems to be needed.

2. Integer-Ratio Policies

For that reason we now introduce the class of integer-ratio policies and show how to compute their average costs. We use the term *facility* to refer to either the warehouse or a retailer. Facility 0 is the warehouse and the retailers are facilities 1 through N.

Let R be an infinite subset of the set of all positive integers and their reciprocals that has no least or greatest element. For each $r \in R$ let r_- be the largest element of R that is smaller than r and let r_+ be the smallest element of R that is larger than r. We require that $1 \in R$ and that $r/r_- \leqslant 2$ for all $r \in R$. This implies that $1/2$ and 2 are in R. For example, R might be the set of all positive integers and their reciprocals, or the set of integer powers of two.

An *integer-ratio* policy is one in which the zero-inventory-ordering property holds, each facility n places an order once every $T_n > 0$ units of time beginning at time $t = 0$, and the ratio T_n/T_0 is in R for all retailers n. The term integer-ratio reflects the fact that either T_n/T_0 or T_0/T_n is an integer. For notational convenience we often denote the order interval T_0 at the warehouse by T.

An integer-ratio policy is stationary-nested if $T_n \leqslant T$ for all retailers n. Stationary-nested policies have an important advantage over integer-ratio policies. In stationary-nested policies the order quantities at the warehouse are stationary. They are periodic in nonnested integer-ratio policies. For example suppose that there are two retailers, their demand rates are both one, $T = 1$, $T_1 = 1/2$, and $T_2 = 2$. The order quantities at the warehouse are $3, 1, 3, 1, \ldots$. The order quantities at the retailers are stationary by the zero-inventory-ordering property.

A *set-up cost* $K_n > 0$ is incurred for every order placed at facility n. For notational reasons it is convenient to think of the system inventory as being composed of N different products with each product n stocked by retailer n and by no others. Inventories of all products can be held simultaneously at the warehouse.

Without loss of generality we choose the unit of each product so as to make the *demand rate* for the product, i.e., the demand rate per unit time at each retailer, equal to two. Let $h'_n > 0$ be the unit *holding cost rate* per unit time at retailer n and let $h^n > 0$ be the unit *holding cost rate* per unit time for product n at the warehouse. Because of the choice of units, the holding cost rate at the warehouse may be different for each product.

The *echelon holding cost rate* at retailer n is $h_n \equiv h'_n - h^n > 0$. The assumption $h_n > 0$ is not essential, but it makes the exposition somewhat easier. Furthermore, the assumption is a natural one because the holding cost rates at retailers normally exceed those at the warehouse. The other case is discussed in Appendix A.

The *average cost* of the integer-ratio policy $\overline{T} = (T, T_1, \ldots, T_N)$ can be written as $c(\overline{T}) \equiv K_0/T + \sum_{n>1} c_n(T, T_n)$ where $c_n(T, T_n)$ is the average cost per unit time of supplying the demand for product n. The average cost c_n includes the setup costs and holding costs at retailer n, and the cost of holding product n at the warehouse. We will show that

$$c_n(T, T_n) = K_n/T_n + h_n T_n + h^n(T \vee T_n), \tag{1}$$

where $\vee$ denotes the maximum. Note that c_n is convex on the positive orthant and that c is strictly convex thereon.

Case 1. *Retailer Order Interval Greater Than Warehouse Order Interval.* When $T_n \geqslant T$ the warehouse places an order simultaneously with every order placed at retailer n. Therefore no inventory of product n is held at the warehouse, and the only costs to consider are those incurred at the retailer. As in the familiar one-stage economic-order-quantity model,

$$c_n(T, T_n) = \left[K_n + (h^n + h_n)T_n^2\right]/T_n = K_n/T_n + (h^n + h_n)T_n.$$

This average cost function leads to the Wilson square-root formula. It is clearly equivalent to (1) when $T_n \geqslant T$.

Case 2. *Retailer Order Interval Less Than Warehouse Order Interval.* When $T_n \leqslant T$ it is convenient to use the echelon method of computing holding costs (Clark and Scarf 1960). The *system inventory* of product n is the sum of the inventory of product n at the warehouse and the inventory at retailer n. It is well known that the system inventory follows the familiar sawtooth inventory pattern with an order interval of T (Graves and Schwarz 1977). The average holding cost of product n is the product of the average system inventory of product n and its holding cost rate at the warehouse, plus the product of the average inventory at retailer n and the echelon holding cost rate there. The average cost is thus $c_n(T, T_n) = K_n/T_n + h_n T_n + h^n T$, which is equivalent to (1) when $T_n \leqslant T$.

3. A Lower Bound on the Average Cost of All Policies by Relaxation

We now drop the integer-ratio constraint $T_n/T \in R$ and consider the relaxed problem of minimizing the average cost $c(\overline{T})$ over all strictly positive vectors $\overline{T}$. The reason for doing this is that the minimum—clearly a lower bound on the average cost of all integer-ratio policies—is in fact a lower bound on the average cost of *all* policies over the infinite horizon. Moreover we show how to round off the optimal relaxed order intervals to obtain an integer-ratio policy with high guaranteed effectiveness.

We first minimize $c_n(T, T_n)$ over all $T_n > 0$ for fixed $T > 0$. Note that $\tau'_n \equiv [K_n/(h^n + h_n)]^{0.5}$ (resp., $\tau_n \equiv (K_n/h_n)^{0.5}$) is the order interval at retailer n given by the Wilson square-root formula with set-up cost K_n, demand rate two, and holding cost rate $(h^n + h_n)$ (resp., h_n). If $G(T) \equiv \{n : T < \tau'_n\}$, $E(T) \equiv \{n : \tau'_n \leqslant T \leqslant \tau_n\}$, and $L(T) \equiv \{n : \tau_n < T\}$, it is easily verified that

$$b_n(T) \equiv \inf_{T_n > 0} c_n(T, T_n) = \begin{cases} 2\left[K_n(h^n + h_n)\right]^{0.5}, & n \in G(T), \\ K_n/T + (h^n + h_n)T, & n \in E(T), \\ 2(K_n h_n)^{0.5} + h^n T, & n \in L(T). \end{cases} \tag{2}$$

The value of T_n that minimizes $c_n(T, T_n)$ is τ'_n if $n \in G(T)$, T if $n \in E(T)$, and τ_n if $n \in L(T)$. Note that $b_n(T)$ is convex and continuously differentiable.

Let $B(T) \equiv K_0/T + \sum_{n \geqslant 1} b_n(T)$ and let T^* minimize $B(T)$ on the positive half-line. Since $b_n(T)$ is convex and continuously differentiable, so is $B(T)$. In fact $B(T)$ is strictly convex, and $B(T) \to \infty$ as $T \to \infty$ and as $T^{-1} \to \infty$. Therefore $B(T)$ attains its minimum at the unique positive number T^* satisfying $B'(T^*) = 0$.

Clearly $B(T)$ is of the form $K(T)/T + M(T) + H(T)T$ where $K(T)$, $M(T)$, and $H(T)$ are piecewise-constant functions of T. Thus $T = T^*$ if and only if $T = [K(T)/H(T)]^{0.5}$. The values of $K(T)$, $M(T)$, and $H(T)$ change when T crosses a τ'_n or a τ_n. If T moves from right to left across τ_n (resp., τ'_n), retailer n moves from $L(T)$ to $E(T)$ (resp., from $E(T)$ to $G(T)$). These $2N$ "breakpoints" give rise to $2N + 1$ "pieces" inside of which $K(T)$, $H(T)$, and $M(T)$ are constant. We can minimize $B(T)$ by beginning with the right-most piece, the one in which T is larger than the largest breakpoint, and moving left from piece to piece until we find the one containing T^*. An algorithm that does this is presented in Appendix A, and we show there that the method can be implemented in $O(N \log N)$ time.

Let $B^* \equiv B(T^*)$. Clearly B^* is a lower bound on the average cost of all integer-ratio policies. We will show that B^* is in fact a lower bound on the average cost of *all* feasible policies. Since the proof of this result uses notation introduced in §4, it is deferred until §5.

4. The Average Cost of Order-Preserving Integer-Ratio Policies and the Lower Bound

In this section we define a subclass of the class of integer-ratio policies called *order-preserving* policies. We will show that the average cost of an order-preserving integer-ratio policy can be expressed as a sum of average costs of the single-facility lot-sizing type, and that the lower bound B^* is the sum of the minima of these functions. Therefore the effectiveness of an order-preserving integer-ratio policy can be established by comparing each single-facility average cost to its minimum.

Let T_n^* be the *optimal relaxed order interval* for retailer n, i.e., the value of T_n that minimizes $c_n(T^*, T_n)$. From (2) we see that the retailers naturally fall into three groups, $G \equiv G(T^*)$, $E \equiv E(T^*)$, and $L \equiv L(T^*)$. Note that the optimal relaxed order interval T_n^* is equal to τ'_n and is greater than T^* if $n \in G$, is equal to T^* if $n \in E$, and is equal to τ_n and is less than T^* if $n \in L$.

We say that the vector $\overline{T}$ *preserves the order* of $\overline{T}^*$ or, briefly, is *order-preserving*, if $T_n \geqslant T$ whenever $n \in G$, $T_n = T$ whenever $n \in E$, and $T_n \leqslant T$ whenever $n \in L$. For each order-preserving vector $\overline{T}$ that is positive, $c(\overline{T})$ can be rewritten as a sum

$$c(\overline{T}) = (K/T + HT) + \sum_{n \in E^c} (K_n/T_n + H_n T_n) \tag{3}$$

of average-cost functions of the single-facility lot-sizing type where $E^c \equiv G \cup L$, $K \equiv K_0 + \sum_{n \in E} K_n$, $H \equiv \sum_{n \in E}(h^n + h_n) + \sum_{n \in L} h^n$, $H_n \equiv (h^n + h_n)$ for all $n \in G$, and $H_n \equiv h_n$ for all $n \in L$. Observe that K and H can be thought of respectively as the aggregate set-up cost and the aggregate holding cost rate per unit time associated with the warehouse and those retailers whose order intervals coincide with that at the warehouse. Also, H_n is the average holding cost per unit time associated with retailer $n \in E^c$.

It follows from the definitions of $\overline{T}^*$, G, E, and L that $\overline{T}^*$ minimizes (3) term-by-term. The resulting minimum relaxed average cost is then

$$B^* = c(\overline{T}^*) = M + \sum_{n \in E^c} M_n \tag{4}$$

where $M \equiv 2(KH)^{0.5}$ and $M_n \equiv 2(K_n H_n)^{0.5}$ for $n \in E^c$.

5. The Proof of the Lower Bound Theorem

In this section we prove the Lower Bound Theorem, which states that the minimum relaxed average cost B^* is a lower bound on the average cost of all feasible policies. The proof is based on the idea that it is possible to allocate the costs incurred by an arbitrary policy to individual facilities in such a way that

- the average cost incurred by the policy is greater than the sum of the average costs allocated to the individual facilities.
- the total average cost allocated to each individual facility is the average cost of a solution to a single-product problem, and
- the sum of the costs of the optimal solutions to these single-product problems is greater than B^*.

The allocation of holding costs to individual facilities is guided by *reallocated holding cost rates*. For each facility $n \in W \equiv E \cup \{0\}$, we define the reallocated holding cost rate H_n by $K_n/H_n = T^{*2}$. For all retailers $n \notin W$, H_n has already been defined.

LEMMA 1 (Reallocated Holding Cost Rates). *Then aggregate holding cost rate H is the sum of the holding cost rates allocated to the individual facilities in W, i.e., $H = \sum_{n \in W} H_n$. Also $B^* = \sum_{n \geqslant 0} M_n$ where M_n is the minimum of $K_n/x + H_n x$ over all $x > 0$. Finally, we have $h_n \leqslant H_n \leqslant h^n + h_n$ for each retailer n, and $H_0 = \sum_{n \geqslant 1} H^n$ where $H^n \equiv h^n + h_n - H_n$ for all retailers n.*

PROOF. Since $K_n/H_n = K/H = T^{*2}$ for all $n \in W$ we have $H = K/T^{*2} = \sum_{n \in W} K_n/T^{*2} = \sum_{n \in W} H_n$. The second assertion follows from (4) and the fact that $M = 2K/T^* = \sum_{n \in W} 2K_n/T^* = \sum_{n \in W} M_n$. The third assertion follows from the fact that $H_n = h_n$ for all $n \in L$, $H_n = h^n + h_n$ for all $n \in G$, and $\tau_n' \leqslant T^* \leqslant \tau_n$ for all $n \in E$, and from the definition of H. Q.E.D.

We are now ready to prove the Lower Bound Theorem.

THEOREM 1 (Lower Bound Theorem). *The minimum relaxed average cost B^* is a lower bound on the average cost of all feasible policies over every finite horizon.*

PROOF. Consider an arbitrary policy over the infinite horizon. Let $C(t')$ be the average cost incurred over the interval $[0, t')$ by this policy. It suffices to show that $C(t') \geqslant B^*$ for all $t' > 0$.

Let J_n be the number of orders placed by facility n in $[0, t')$, I_n^t be the inventory at retailer n at time t, $S_n^t \geqslant I_n^t$ be the system inventory of product n at time t, and $I_0^t \equiv \sum_{n \geqslant 1} H_0^{-1} H^n S_n^t$ be the average system inventory at time t, averaged over all products. Observe that I_n^t is right-continuous in t, has jumps (upward) at the times at which facility $n \geqslant 0$ orders, and decreases linearly in t with slope -2 otherwise.

Using Lemma 1, we allocate the total holding cost incurred in $[0, t')$ to individual facilities as follows:

$$\sum_{n\geqslant 1}\int_0^{t'}(h_nI_n^t+h^nS_n^t)\,dt\geqslant\sum_{n\geqslant 1}\int_0^{t'}(H_nI_n^t+H^nS_n^t)\,dt=\sum_{n\geqslant 0}\int_0^{t'}H_nI_n^t\,dt.$$

Now the nth term in the sum on the right-hand side of the above equality is the total holding cost incurred in $[0,t')$ in a *single-item* economic-lot-size problem in which J_n orders are placed in $[0,t')$, the setup cost is K_n, the demand rate per unit time is two, and the unit holding cost per unit time is H_n. It is well known (Carr and Howe 1962) that the minimum-cost inventory policy for this problem among those in which J_n orders are placed in the interval $[0,t')$ entails ordering $2x_n$ units every $x_n\equiv t'/J_n$ units of time with the resulting total holding cost $t'H_nx_n$. Thus by Lemma 1,

$$C(t')\geqslant\sum_{n\geqslant 0}\left[K_nJ_n+\int_0^{t'}H_nI_n^t\,dt\right]\Big/t'\geqslant\sum_{n\geqslant 0}\left[K_n/x_n+H_nx_n\right]\geqslant B^*.\quad \text{Q.E.D.}$$

6. q-Optimal Integer-Ratio Lot-Sizing

In this section we show how to find an integer-ratio policy whose effectiveness is at least 94%. We define the *fractional effectiveness* of a policy to be the ratio of the infimum of the average cost over all policies to the average cost of the policy in question. Thus a $100e$%-effective policy has fractional effectiveness e.

In order to find a lower bound on the fractional effectiveness of an order-preserving integer-ratio policy, it is convenient to write (3) in an alternate form. To that end, let $e(\alpha)\equiv 2/(\alpha+\alpha^{-1})$ for $\alpha>0$. Then since $K/T^*=HT^*=M/2$ and $K_n/T_n^*=H_nT_n^*=M_n/2$ for retailers $n\in E^c$, it follows from (3) that

$$c(\bar{T})=M/e(q)+\sum_{n\in E^c}M_n/e(q_n) \tag{5}$$

where $q\equiv T/T^*$ and $q_n\equiv T_n/T_n^*$. Observe that if $\bar{T}=\bar{T}^*$ then (5) reduces to (4) because $e(1)=1$. Also note that if $\bar{T}$ is an order-preserving integer-ratio policy, then $e(q)$ is the fractional effectiveness of $\bar{T}$ at the warehouse and at each retailer combined therewith. Similarly, $e(q_n)$ is the fractional effectiveness of $\bar{T}$ at retailer $n\in E^c$. Thus *the effectiveness of $\bar{T}$ at each facility depends only on the quotients q, q_n of the order intervals that $\bar{T}$ and $\bar{T}^*$ use there, and is otherwise independent of the cost and demand data*. For this reason we will call q and q_n *effectiveness quotients*.

It follows from (5) and the Lower Bound Theorem that the fractional effectiveness of an order-preserving integer-ratio policy $\bar{T}$ is at least

$$\frac{c(\bar{T}^*)}{c(\bar{T})}=\left[(M/B^*)/e(q)+\sum_{n\in E^c}(M_n/B^*)/e(q_n)\right]^{-1}$$
$$\geqslant e(q)\wedge\left[\min_{n\in E^c}e(q_n)\right] \tag{6}$$

where $\wedge$ denotes the minimum. Thus a sufficient condition for $\bar{T}$ to have effectiveness at least ϵ, say, is that $\bar{T}$ have at least that effectiveness at each facility.

A policy is said to be *q-optimal* if it is optimal among all integer-ratio policies for which $T=qT^*$. We will show how to find a q-optimal policy that is order-preserving and whose fractional effectiveness at each individual facility is at least $e(\sqrt{2})\approx 0.94$. To that end, note that $e(\alpha)$ is strictly quasiconcave, achieves its maximum at $\alpha=1$, and satisfies $e(\alpha)=e(1/\alpha)$ for all $\alpha>0$. Therefore the fractional effectiveness $e(q)$ (resp., $e(q_n)$) at the warehouse (resp., retailer $n\in E^c$) is at least as large as $e(\sqrt{2})$ if and only if q (resp., q_n) is in the *root-two interval* $[\sqrt{0.5},\sqrt{2}]$.

We denote the retailer-to-warehouse *order-interval ratio* for retailer n by $r_n \equiv T_n/T \in R$. Note that an integer-ratio policy $\overline{T}$ is uniquely determined by T and by the r_n, $1 \leqslant n \leqslant N$.

We will show that if q is in the root-two interval there is always a q-optimal integer-ratio policy that is order-preserving. Therefore a q-optimal policy can be found by choosing for each $n \in E^c$ an order-interval ratio r_n that maximizes the retailer's fractional effectiveness $e(q_n)$ subject to $r_n \in R$, $r_n \geqslant 1$ for all $n \in G$, and $r_n \leqslant 1$ for all $n \in L$.

LEMMA 2 (Existence and Characterization of Order-Preserving q-Optimal Integer-Ratio Policies). *For every warehouse effectiveness quotient q in the root-two interval, there is a q-optimal integer-ratio policy that is order-preserving. Furthermore $\overline{T}$ is such a policy if and only if $r_n = 1$ for all $n \in E$, and*

$$r_n = r \qquad \textit{for some} \quad r \in R \quad \textit{with} \quad (rr_-)^{0.5} \leqslant T_n^*/T \leqslant (rr_+)^{0.5} \tag{7}$$

for all $n \in E^c$.

PROOF. We first show that there is such a policy. It suffices to show that there is a q-optimal integer-ratio policy satisfying $r_n \geqslant 1$ for all $n \notin L$ and $r_n \leqslant 1$ for all $n \notin G$. If $n \notin L$ then $T/\tau_n \leqslant T/T^* = q \leqslant \sqrt{2}$, so by (1) and the fact that $\tau_n^2 = K_n/h_n$,

$$c_n(T, 0.5T) - c_n(T,T) = (\tau_n^2 - 0.5T^2)h_n/T \geqslant 0.$$

Thus since c_n is convex and r_n cannot lie strictly between 0.5 and 1 because either r_n or $1/r_n$ must be an integer, one r_n that minimizes $c_n(T, r_nT)$ over all $r_n \in R$ satisfies $r_n \geqslant 1$. Similarly, if $n \notin G$, then $T/\tau_n' \geqslant T/T^* = q \geqslant \sqrt{0.5}$, so

$$c_n(T, 2T) - c_n(T,T) = (T^2 - 0.5\tau_n'^2)(h^n + h_n)/T \geqslant 0$$

and, as above, one r_n that minimizes $c_n(T, r_nT)$ over R satisfies $r_n \leqslant 1$. This establishes the existence of an order-preserving, q-optimal integer-ratio policy.

To complete the proof it suffices to show that for each $n \in E^c$, r_n maximizes $e(q_n)$ subject to $r_n \in R$ if and only if (7) holds, and that in that event $r_n \geqslant 1$ if $n \in G$ and $r_n \leqslant 1$ if $n \in L$.

To that end, on putting $\rho \equiv T_n^*/T$ and $r = r_n$, $e(q_n) = e(r/\rho)$. We will show that r satisfies (7) if and only if r maximizes $e(r/\rho)$ over R, or since e is strictly quasiconcave on the positive real line, $e(r/\rho) \geqslant e(r_-/\rho) \vee e(r_+/\rho)$. Now since $e(\alpha) \geqslant e(\beta)$ if and only if $\alpha \vee \alpha^{-1} \leqslant \beta \vee \beta^{-1}$ for $\alpha, \beta > 0$, the above inequality holds if and only if

$$r/\rho \vee \rho/r \leqslant (r_-/\rho \vee \rho/r_-) \wedge (r_+/\rho \vee \rho/r_+).$$

Since $r_- \leqslant r \leqslant r_+$, this is so if and only if $r/\rho \leqslant \rho/r_-$ and $\rho/r \leqslant r_+/\rho$, or equivalently, (7) holds.

It remains to show that if (7) holds, then $r_n \geqslant 1$ for all $n \in G$ and $r_n \leqslant 1$ for all $n \in L$. If $n \in G$, $T_n^*/T = \tau_n'/T > q^{-1} \geqslant \sqrt{0.5} = (1 \cdot 1_-)^{0.5}$, so by (7), $r_n \geqslant 1$. If $n \in L$, $T_n^*/T = \tau_n/T < q^{-1} \leqslant \sqrt{2} = (1 \cdot 1_+)^{0.5}$, so $r_n \leqslant 1$. This concludes the proof of Lemma 2. Q.E.D.

THEOREM 2 (94%-Effectiveness Theorem). *There is a q-optimal integer-ratio policy with effectiveness at least* $200\%/(\sqrt{2} + \sqrt{0.5}) > 94\%$.

PROOF. By (6), the Lower Bound Theorem, and Lemma 2, it suffices to show that for q-optimal integer-ratio policies, the fractional effectiveness for each individual facility is at least $e(\sqrt{2}) = 2/(\sqrt{2} + \sqrt{0.5})$. Recall that for all $r \in R$, $0.5 \leqslant r_-/r$ and $r_+/r \leqslant 2$. Dividing (7) by r_n we have $\sqrt{0.5} \leqslant T_n^*/T_n = q_n^{-1} \leqslant \sqrt{2}$ for all $n \in E^c$. Since

$\alpha, \beta > 0$ and $\alpha \vee \alpha^{-1} \leqslant \beta \vee \beta^{-1}$ imply $e(\alpha) \geqslant e(\beta)$, $e(q_n) \geqslant e(\sqrt{2})$. Similarly, because q is in the root-two interval, $e(q) \geqslant e(\sqrt{2})$. Q.E.D.

This establishes the worst-case effectiveness of a q-optimal integer-ratio policy. The average effectiveness will be considerably higher than 94%. In fact, once an integer-ratio policy $\overline{T}$ has been computed, an ex-post lower bound $100\% \times B^*/c(\overline{T})$ on the effectiveness of $\overline{T}$ is available.

An important advantage of q-optimal lot-sizing is that it allows considerable freedom in choosing the order intervals. If the order interval T at the warehouse is chosen to be a convenient time period, such as three months, the order intervals for the retailers will probably be convenient as well. We can insure that this will be the case by deferring the choice of R until after T has been chosen. For example, if T is taken to be twelve weeks we may want to include $1/3$ (four weeks) and $1/6$ (two weeks) in R, but not to include $1/4$ (three weeks) or $1/5$ (2.4 weeks) therein.

7. Optimal Power-of-Two Lot-Sizing

It is of interest to find easily-computed policies whose guaranteed effectiveness exceeds 94%—the best possible for q-optimal integer-ratio policies. One way to do this would be to compute an optimal integer-ratio policy. There are several methods available for computing optimal stationary-nested policies and similar methods could be used for computing optimal integer-ratio policies (Goyal 1974a, Graves and Schwarz 1977, Singer 1978). The worst-case running times of these algorithms are at least exponential, but the algorithms are reasonably straightforward and will certainly work well on the average. We will explore another alternative that leads to a much simpler algorithm that finds a 98%-effective policy in $O(N \log N)$ time.

The effectiveness of a q-optimal integer-ratio policy is at least $100e(\sqrt{2})\% \approx 94\%$ because $r/r_- \leqslant 2$ for all $r \in R$. One way of making q-optimal integer-ratio policies more effective would be to reduce the maximum value of r/r_-; but for integer-ratio policies this is impossible because R must contain one and there are no integers between one and two. The fact that the worst-case effectiveness of a q-optimal integer-ratio policy depends only on the maximum value of r/r_- leads us to consider the set R for which $r/r_- = 2$ for *all* ratios $r \in R$, namely, the set of integer powers of two. Integer-ratio policies for this set R are called *power-of-two* policies.

We will restrict attention to power-of-two policies in this section. While this is done primarily for computational reasons, the simplicity of power-of-two policies is very attractive from a managerial viewpoint as well. For example, suppose that the order intervals of all facilities are between three days and six months. A power-of-two policy can have at most seven distinct order intervals, regardless of the number of retailers.

Of course q-optimal power-of-two policies are 94% effective for all q in the root-two interval. This is a wide interval whose upper bound is double its lower bound. It is reasonable to hope that the effectiveness can be improved if we optimize over q as well. This turns out to be the case. Indeed, in this section we show how to find an optimal power-of-two policy in $O(N \log N)$ time and prove that this policy is at least 98% effective.

Overview of the Method for Computing Optimal Power-of-Two Policies

In our discussion of power-of-two policies we will use the following definitions. The *ratio set* determined by an order-preserving power-of-two policy $\overline{T}$ is the set of order-interval ratios $\{r_n : n \in E^c\}$. An order-preserving power-of-two policy is completely determined by q and its ratio set. A ratio set is called *efficient* if it is determined by an order-preserving q-optimal power-of-two policy for some q in the root-two interval.

We will show that q is in the root-two interval for all optimal power-of-two policies. This will imply by Lemma 2 that there is an optimal power-of-two policy that is order-preserving. Such a policy must determine an efficient ratio set. We then show that there are only $|E^c| + 1$ efficient ratio sets, and that they can easily be identified. Moreover, given any efficient ratio set we can compute an optimal value of q. This gives us $|E^c| + 1$ policies, the best of which is an optimal power-of-two policy. Finally, we prove that this policy is at least 98% effective.

This method of finding an optimal power-of-two policy is similar to a method devised by Goyal (1974b) for the joint-replenishment problem. The relationships between the two methods are discussed at the end of the section.

We begin with the following lemma, which is proved in Appendix B.

LEMMA 3 (Optimal Power-of-Two Policies Have Effectiveness Quotients in the Root-Two Interval). *In an optimal power-of-two policy, the effectiveness quotients of all facilities lie in the root-two interval.*

Lemma 3 is not valid for integer-ratio policies in which the order-interval ratios r_n are not restricted to powers of two. For when the costs at the warehouse are relatively low, it may be advantageous to move q out of the root-two interval in order to allow the order intervals for the retailers to be closer to their optimal relaxed values. An example in Appendix B illustrates this fact. By Lemmas 2 and 3, there is an optimal power-of-two policy that is order-preserving. We now wish to describe such a policy as a function of q, $\sqrt{0.5} \leqslant q \leqslant \sqrt{2}$. Let $r_n^* \equiv T_n^*/T^*$ be the *optimal relaxed-order-interval ratio* at retailer n, let p_n be the integer determined by $r_n^* \leqslant 2^{p_n} < 2r_n^*$, and let $Q_n \equiv r_n^* 2^{0.5-p_n}$ for all $n \in E^c$.

LEMMA 4 (Characterization of Order-Preserving q-Optimal Power-of-Two Policies). *A power-of-two policy $\overline{T}$ is order-preserving and q-optimal if and only if $r_n = 1$ for all $n \in E$ and q and q_n are in the root-two interval for each $n \in E^c$. Moreover an optimal choice for r_n, $n \in E^c$ is given by*

$$r_n = \begin{cases} 2^{p_n} & \text{for} \quad q < Q_n, \\ 2^{p_n - 1} & \text{for} \quad q \geqslant Q_n. \end{cases} \tag{8}$$

PROOF. Note that $r/r_- = r_+/r = 2$ for all $r \in R$. For each $n \in E^c$, on setting $T = T_n/r$ in (7), Lemma 3 implies that in an order-preserving, q-optimal power-of-two policy, q_n is in the root-two interval if and only if $r = r_n$ is q-optimal.

The effectiveness quotient for retailer $n \in E^c$ can be written as $q_n = r_n q/r_n^*$. It is easily verified that if we choose $r_n = 2^{p_n}$ if $q < Q_n$ and $r_n = 2^{p_n - 1}$ if $Q_n \leqslant q$, then q_n lies in the root-two interval. Q.E.D.

Computing an Optimal Power-of-Two Policy

Let $Q_{[m]}$ be the mth smallest of the Q_n, $n \in E^c$, and let $Q_{[0]} \equiv \sqrt{0.5}$ and $Q_{[M+1]} \equiv \sqrt{2}_+$ where $M = |E^c|$. Then if $Q_{[m]} \leqslant q < Q_{[m+1]}$, by Lemma 5 the ratio set

$$r_n = 2^{p_n - 1} \quad \text{if} \quad Q_n \leqslant Q_{[m]} \quad \text{and} \quad r_n = 2^{p_n} \quad \text{if} \quad Q_{[m]} < Q_n \tag{9}$$

is efficient. For each m, $0 \leqslant m \leqslant M$, we will refer to the set of retailer-to-warehouse order-interval ratios given by (9) as the *mth efficient ratio set*. By Lemmas 3 and 4, one of these efficient ratio sets is determined by an optimal power-of-two policy.

Denote by $(T:m)$ the order-preserving power-of-two policy in which the order interval at the warehouse is $T > 0$ and the retailer-to-warehouse order-interval ratios are given by the mth efficient ratio set, $m = 0, \ldots, M$. Observe that $(T:m)$ is order-preserving even when T lies outside of the root-two interval. Moreover $T_n = r_n T$ by (3), and by (9), the average cost incurred by $(T:m)$ is $c(T:m) = K^m/T + H^m T$ for all $T > 0$ where

$$K^m \equiv K + \sum_{Q_n \leqslant Q_{[m]}} K_n 2^{1-p_n} + \sum_{Q_n > Q_{[m]}} K_n 2^{-p_n} \quad \text{and}$$

$$H^m \equiv H + \sum_{Q_n \leqslant Q_{[m]}} H_n 2^{p_n - 1} + \sum_{Q_n > Q_{[m]}} H_n 2^{p_n}.$$

For fixed m, the value of T that minimizes $c(T:m)$ on the positive half-line is $T^m = (K^m/H^m)^{0.5}$ and the corresponding minimum average cost is $C_m \equiv 2(K^m H^m)^{0.5}$. Since this class of policies contains an optimal power-of-two policy, one optimal power-of-two policy is $(T' : m')$, which is obtained by choosing $m = m'$ to minimize C_m and setting $T' = T^{m'}$.

THEOREM 3 (98%-Effectiveness Theorem). *An optimal power-of-two policy has effectiveness at least* $\sqrt{2}\ln 2 \times 100\% > 98\%$.

PROOF. For each q in the root-two interval, let $C(q)$ denote the average cost of an order-preserving q-optimal power-of-two policy, which exists by Lemma 2. The average cost of an optimal power-of-two policy is the minimum of $C(q)$ and is attained in the root-two interval by Lemma 3. It remains to show that such a policy is at least 98% effective.

We will use as an upper bound on $\min_{q>0} C(q)$ the weighted average of $C(q)$ over the root-two interval, weighted with respect to the density function $1/(q \ln 2)$. Recall from (5) that $C(q)$ is the sum of $M/e(q)$ and $M_n/e(q_n)$ for $n \in E^c$ where q_n is a q-optimal effectiveness quotient for retailer n. We will show that the weighted average of $1/e(q_n)$ as q runs over the root-two interval is equal to the weighted average of $1/e(q)$ and is thus independent of the input data. In fact the density function $1/(q \ln 2)$ was chosen to make this property hold.

It follows from Lemma 4 that for each retailer $n \in E^c$, $q_n = \sqrt{2}\, q/Q_n$ if $q < Q_n$ and $q_n = \sqrt{0.5}\, q/Q_n$ if $q > Q_n$. Therefore

$$\int_{\sqrt{0.5}}^{\sqrt{2}} 1/e(q_n) \frac{dq}{q \ln 2}$$

$$= \int_{1/Q_n}^{\sqrt{2}} 1/e(\alpha) \frac{d\alpha}{\alpha \ln 2} + \int_{\sqrt{0.5}}^{1/Q_n} 1/e(\alpha) \frac{d\alpha}{\alpha \ln 2}$$

$$= \int_{\sqrt{0.5}}^{\sqrt{2}} 1/e(q) \frac{dq}{q \ln 2} = \frac{1}{\ln 2} \int_{\sqrt{0.5}}^{\sqrt{2}} \left(\frac{1}{2q^2} + \frac{1}{2} \right) dq = \frac{1}{\sqrt{2} \ln 2}.$$

Therefore by (5) and (4),

$$\min_{\sqrt{0.5} \leqslant q \leqslant \sqrt{2}} C(q) \leqslant \int_{\sqrt{0.5}}^{\sqrt{2}} C(q) \frac{dq}{q \ln 2} = \frac{B^*}{\sqrt{2} \ln 2}. \quad \text{Q.E.D.}$$

We can calculate K^0 and H^0 in $O(N)$ time using (9), and given K^{m-1} and H^{m-1} we can calculate K^m and H^m in constant time. The hardest part of optimal power-of-two lot-sizing is sorting the Q_n, which requires $O(N \log N)$ comparisons. Thus the overall running time for computing an optimal power-of-two policy is $O(N \log N)$.

As was mentioned, Goyal used an algorithm similar to this one to find an optimal policy for the joint-replenishment problem (Goyal 1974b), a special case of the one-warehouse, multi-retailer system. However, because he does not restrict his order intervals to being powers of two, the analog of Lemma 3 does not hold. Indeed, for the two-retailer case it is possible to adjust the data of the problem so as to make the number of arithmetic operations required by his algorithm arbitrarily large. Power-of-two order intervals allow us to obtain an extremely efficient algorithm which, although it may not find an optimal integer-ratio policy, does produce a policy having an average cost within 2% of the minimum possible.

8. Conclusions

Both q-optimal integer-ratio and optimal power-of-two lot-sizing are very efficient. The most important advantage of q-optimal integer-ratio lot-sizing is the flexibility it allows in choosing the order intervals to correspond to easily-implemented time periods.

It is likely that by using a somewhat more general class of policies, such as allowing $r_n = 2/3$ or $r_n = 3/2$, or by finding an optimal integer-ratio policy, the worst-case effectiveness of the lot-sizing rules given herein could be improved. However in most applications the errors inherent in the model and in the data will be more significant than the 2% worst-case difference between the average cost of an optimal power-of-two policy and the lower bound.

Williams has discussed and tested a number of heuristic algorithms which take an arbitrary stationary-nested policy and try to improve on it (Crowston, Wagner and Henshaw 1972, Graves and Schwarz 1977, Williams 1981). These algorithms can easily be adapted to the larger class of integer-ratio policies. They often improve on q-optimal integer-ratio policies and, if ratios other than powers of two are used, on optimal power-of-two policies. They could also be used independently of the policies developed in this paper, but it is not clear what their worst-case running time or effectiveness would be.

The ease of computing q-optimal integer-ratio and optimal power-of-two policies and their effectiveness and simplicity make them very attractive. Extensions to multi-product, multi-stage production/inventory systems are found in Roundy (1984).[1]

Appendix A. The Algorithm for Minimizing the Relaxed Average Cost

In this appendix we formally present the algorithm for minimizing $B(T)$, analyze its efficiency, and describe the modifications necessary if there are retailers with nonpositive echelon holding costs. We begin with the algorithm.

The Minimum-Relaxed-Average-Cost Algorithm

Step 1. *Calculate and Sort the Breakpoints.* Calculate the breakpoints $\tau_n' = [K_n/(h^n + h_n)]^{0.5}$ and $\tau_n = (K_n/h_n)^{0.5}$ and sort them to form a nondecreasing sequence of $2N$ numbers. Label each breakpoint with the value of n and with an indicator showing whether it is the left breakpoint τ_n' or the right breakpoint τ_n.

Step 2. *Initialize E, G, L, K, and H.* Set $E = G = \emptyset$, $L = \{1, \ldots, N\}$, $K = K_0$, and $H = \sum_{n \geqslant 1} h^n$.

[1] I am especially indebted to Arthur F. Veinott, Jr. He called the problem to my attention, and the exposition has benefitted greatly from his comments. In particular, the elegance of the proof of the Lower-Bound Theorem given herein is due to him. He also made significant improvements in my proofs of Lemmas 2 and 4. I am also indebted to Professor Daniel Granot for the helpful comments he made on an earlier version of the paper.

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Step 3. *Cross the Largest Uncrossed Breakpoint*. Let τ be the largest previously uncrossed breakpoint. If $\tau^2 \geqslant K/H$ and $\tau = \tau_n$ is a right breakpoint, cross τ and update E, L, K, and H by $E \leftarrow E \cup \{n\}$, $L \leftarrow L\backslash\{n\}$, $K \leftarrow K + K_n$ and $H \leftarrow H + h_n$. Then go to Step 3. If $\tau^2 > K/H$ and $\tau = \tau_n'$ is a left breakpoint, cross τ and update E, G, K, and H by $E \leftarrow E\backslash\{n\}$, $G \leftarrow G \cup \{n\}$, $H \leftarrow H - h^n - h_n$ and $K \leftarrow K - K_n$, and go to Step 3. Otherwise T^* is in the current piece. Go to Step 4.

Step 4. *Calculate* T^* *and* B^*. Set $T^* = (K/H)^{0.5}$. Then T_n^* for all retailers $n \in E^c$ and B^* can be calculated as in §4.

It remains to be shown that Step 3 will be executed before the last breakpoint is crossed. Otherwise we would have $H = 0$ and $E \cup L = \emptyset$. If $\tau = (K_n/h_n')^{0.5}$ is the only uncrossed breakpoint, it is a left breakpoint. Then in Step 3 $K = K_0 + K_n$ and $H = h_n'$, so $K/H > \tau^2$. Therefore Step 4 will be executed and the algorithm will terminate.

Running Time of the Relaxed-Average-Cost Algorithm

The sort in Step 1 requires $O(N \log N)$ comparisons. All other phases of the algorithm require a number of operations that is linear in N. Only N square roots are needed if we use $\tau_n'^2$ and τ_n^2 instead of τ_n' and τ_n in Steps 1 and 3.

Retailers with Nonpositive Incremental Holding Costs

Suppose that retailer n has a nonpositive incremental holding cost. Since it is as economical to hold goods at the retailer as it is at the warehouse, we would expect $T_n^* \geqslant T^*$. To see that this is indeed the case, note that $c_n(T^*, T_n)$ is decreasing in $T_n < T^*$. Since c_n is convex and $T_n = T_n^*$ minimizes $c_n(T^*, T_n)$ over all $T_n > 0$, $T_n^* \geqslant T^*$. Thus $n \in E \cup G$. Note that τ_n does not exist, but if we consider it to be infinite wherever it appears, the proofs of all lemmas and theorems given herein remain valid.

The following modifications are made to the minimum-relaxed-average-cost algorithm. In Step 1 we calculate only the left breakpoint for retailer n. In Step 2 we place n in E rather than L, we add K_n to K, and we add $h^n + h_n$ to H in place of h^n.

Appendix B. The Proof of Lemma 3

In this appendix we prove Lemma 3. We also show by example that this lemma does not hold unless we restrict ourselves to power-of-two policies.

PROOF. Given an arbitrary power-of-two policy $\bar{T}'$, we will find an order-preserving power-of-two policy $\bar{T}$ such that the effectiveness quotients $q_n = T_n/T_n^*$, $n \geqslant 0$, all lie in the root-two interval where $T_n = T$ and $T_n^* = T^*$ for all $n \in W$. The average cost of $\bar{T}$ is shown not to exceed that of $\bar{T}'$, and to be strictly less than that of $\bar{T}'$ if, for some $n \geqslant 0$, $q_n' \equiv T_n'/T_n^*$ is not in the root-two interval. Because $\bar{T}'$ is not necessarily order-preserving, we must compute costs separately for the warehouse and for each of the retailers $n \in E$ combined with the warehouse.

By (1) and Lemma 1

$$c_n(T', T_n') = K_n/T_n' + h_n T_n' + h^n(T' \vee T_n') \geqslant K_n/T_n' + H_n T_n' + H^n T'.$$

Therefore

$$c(\bar{T}') = K_0/T' + \sum_{n \geqslant 1} c_n(T', T_n') \geqslant \sum_{n \geqslant 0} (K_n/T_n' + H_n T_n') = \sum_{n \geqslant 0} M_n/e(q_n').$$

We define q_n for all $n \geqslant 0$ by $q_n \equiv 2^{l_n} q_n'$ where l_n is an integer chosen so that q_n lies in the root-two interval and is less than $\sqrt{2}$. Since $T_n'/T_n = q_n'/q_n$ and T_m'/T_n' are powers of two for all $m, n \geqslant 0$, T_m/T_n must be a power of two as well. Note that $T_m \geqslant T_n$ whenever $T_m^* \geqslant T_n^*$, so $\bar{T}$ is order-preserving.

We now show that $e(q_n) \geqslant e(q_n')$ for all $n \geqslant 0$. Since q_n/q_n' is a power of two for each facility n, either $q_n = q_n'$ holds or $\sqrt{0.5} < q_n' < \sqrt{2}$ does not hold, and in either case $e(q_n) \geqslant e(q_n')$. Therefore

$$c(\bar{T}') \geqslant \sum_{n \geqslant 0} M_n/e(q_n') \geqslant \sum_{n \geqslant 0} M_n/e(q_n) = c(\bar{T}).$$

If q_n' is not in the root-two interval for any $n \geqslant 0$ then $e(q_n) > e(q_n')$, so $c(\bar{T}') > c(\bar{T})$. Q.E.D.

The following example illustrates that Lemma 3 does not hold unless the order-interval ratios are restricted to powers of two.

EXAMPLE (An Optimal Integer-Ratio Policy with Warehouse Effectiveness Quotient Not in the Root-Two Interval). Consider the two-retailer system in which the optimal relaxed order intervals are $T^* = 1$, $T_1^* = 0.9$, and $T_2^* = 1/1.8$ and the minima of the single-facility cost functions are $M = 0.001$, $M_1 = 1$, and $M_2 = 1$. In the optimal integer-ratio policy in which $T = q$ lies in the root-two interval, $T = T_1 = 1$ and $T_2 = 0.5$. By (3) the average cost of this policy is $0.001/e(1) + 1/e(0.9) + 1/e(1/0.9) \simeq 2.01$. However the integer-ratio policy in which $T = 9.9$, $T_1 = 0.9$, and $T_2 = 0.55$ has an average cost of $0.001/e(9.9) + 1/e(1) + 1/e(0.99) \simeq 2.005$.

It is also possible to show that there need not be an optimal integer-ratio policy that is order-preserving.

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